

Worldline as a Spin Chain

Amir H. Fatollahi

*Department of Physics, Alzahra University,
P. O. Box 19938, Tehran 91167, Iran*

fath@alzahra.ac.ir

Abstract

The general theoretical ground for the models based on the compact angle coordinates is presented. It is observed that the proper dependence on compact coordinates has to be through the group elements and is achieved most naturally in a discrete-time formulation of the theory. By the construction, the discrete worldline inlaid by compact coordinates resembles the spin chains of magnetic systems. As examples, the models based on the groups $U(1)$, \mathbb{Z}_N and $SU(2)$ are explicitly constructed and their exact energy spectra are obtained. As the consequence of minima in the spectra, the models exhibit a phase transition of first-order. The dynamics by $U(1)$ group is attempted to be fitted to the proposed role for monopoles in the dual Meissner effect of confinement mechanism.

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1 Introduction

It has been known that treating the gauge fields as compact angle variables would reveal some non-trivial aspects of gauge field theories [1–4]. Among studies based on the compactness of gauge fields, the lattice formulation of gauge theories has provided an explanation for the confinement mechanism as well as a basis for numerical studies at strong coupling regime [4, 5].

By now there are specific instances of affinities between coordinates and gauge fields. The oldest example is the one by the special theory of relativity, by which it is understood that both space-time coordinates and gauge fields transform as 4-vectors under the Lorentz transformations. As another example, based on the duality proposed in [6], one can formulate the coordinate/field correspondence in both Abelian and non-Abelian gauge theories [7]. The other instance of relation between coordinates and gauge fields is provided by T-duality of string theory. Accordingly, the transverse coordinates of Dp-branes in the dual theory are represented by the gauge fields of open string states, leading to the correspondence [8, 9]

$$A_i \longleftrightarrow X_i/l_s^2, \quad (1)$$

in which l_s is the string theory length. At weak coupling the dynamics of X_i 's is captured by the theory resulted from dimensional reduction of the ordinary U(1) gauge theory [8, 9]. In particular the reduction on all spatial components of the gauge field yields the D0-brane dynamics, namely [8]

$$S_{D0} = \int dt \frac{m_0}{2} \dot{x}_i^2, \quad (2)$$

in which $m_0 \propto 1/g^2$ (g : gauge coupling) [8]. The transverse coordinates of N number of Dp-branes are represented by N dimensional hermitian matrices [10].

In [11] the dimensional reduction of pure U(1) lattice gauge theory is considered to model the dynamics of 0-branes at strong coupling regime. The model by [11] might be considered as the result of combination of two themes mentioned earlier, 1) treating gauge fields as compact angle variables [1–4], 2) assuming similar characters between coordinates and gauge fields [6–9]. The explicit form of the action after the dimensional reduction of U(1) lattice gauge theory is:

$$S_0 = \frac{1}{g^2} \sum_n \left(\cos \frac{x_{n+1} - x_n}{R} - 1 \right) \quad (3)$$

in which the coordinates appear as the compact angle variables depending on discrete imaginary time label n . By this form the worldline theory takes the form of the 1D plane-rotator model of spin lattice systems [12]. Based on the prescription for the original lattice gauge theory [4], using the transfer-matrix method the quantization of the model is formulated. The exact energy spectrum as the function of gauge coupling is obtained, with a minimum at critical coupling $g_c = 1.125$ in the lowest energy. As the direct consequence of the minimum, the model exhibits a first-order phase transition between coexistent phases with small and large couplings [11]. Based on discontinuous nature of the first-order phase transition, for $g < g_c$ and $T \approx 0$ the effective zero mean-square velocity $\langle v^2 \rangle$ is zero. The possible relation between the model and the proposed role for magnetic monopoles in confinement mechanism based on dual Meissner effect is discussed [11].

The purpose of the present work is to provide a general theoretical ground for the models based on compact angle coordinates. It is clarified that the dependence on the group elements rather than the algebra ones would lead to invariance of the action under the total shifts in the compact domain, like that is happening in (3). It is observed that the proper dependence is achieved most naturally in a discrete-time formulation of the theory. The worldline action by the formulation resembles the spin chain Hamiltonian of magnetic systems, with coordinates appearing as the spin degrees of freedom.

The transfer-matrix method is used to define the quantum theory [13]. In a path-integral representation of the formulation, it is emphasized that the pre-factor in the definition of the elements of transfer-matrix, in contrast to the case with infinite extent coordinates, can not be absorbed by a change of the integration variables. This particularly causes the spectra developing minima as a function of the defining parameter of the theory.

As examples for the formulation, the models based on the $U(1)$, \mathbb{Z}_N and $SU(2)$ groups are explicitly constructed. In all of the models based on these groups the exact energy eigen-values are obtained. As the consequence of the minima in the spectrum all the models exhibit the first-order phase transitions. The phase transition structures of the models are discussed based on the Gibbs free energy.

The organization of the rest of the paper is as follows. In Sec. 2 the basic assumptions and ingredients for the formulation based on compact coordinates are presented. In Secs. 3, 4 and 5 three examples based on the groups $U(1)$, \mathbb{Z}_N and

SU(2) are presented explicitly. For all the three groups the energy eigen-values are obtained exactly, together with the discussion on the nature of the phase transitions in them. Sec. 6 is devoted to conclusion and discussion.

2 Basics and Formulation

The transition amplitude between positions x_0 and x_N at times t_0 and t_N is represented by the path-integral [14]

$$\langle x_N, t_N | x_0, t_0 \rangle = \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \prod_{n=1}^{N-1} \sqrt{\frac{m}{2\pi i \hbar \epsilon}} dx_n e^{iS[t_0, t_N]/\hbar} \quad (4)$$

in which $\epsilon = (t_N - t_0)/N$, tending to zero in the limit $N \rightarrow \infty$. It is noticed that in above all the intermediate positions x_n 's have infinite extents, $-\infty < x_n < \infty$. From the considerations mentioned earlier in Introduction, here the main concern is about the finite-extent coordinates. The finite-extent coordinates are quite known in physics. The most familiar ones are the angle variables of polar coordinates in 2D and 3D problems. The other example happens when the system is defined inside a finite volume, like a box or a sphere. As a definite example, let us consider a free particle on a circle of radius R . Defining the angle variable $\phi = x/R$ with $-\pi \leq \phi \leq \pi$, we have

$$L = \frac{1}{2} m R^2 \dot{\phi}^2 \rightarrow H = \frac{p_\phi^2}{2mR^2} \quad (5)$$

leading to the eigen-functions and eigen-values as follow

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}, \quad E_n = \frac{n^2 \hbar^2}{2mR^2} \quad (6)$$

Then the transition amplitude between two different positions is known to be

$$\begin{aligned} \langle \phi_N, t_N | \phi_0, t_0 \rangle &= \sum_{n=-\infty}^{\infty} \psi_n(\phi_N) \psi_n^*(\phi_0) e^{-iE_n(t_N - t_0)/\hbar} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\phi_N - \phi_0)} e^{-i n^2 \hbar (t_N - t_0) / 2mR^2} \end{aligned} \quad (7)$$

The above can be expressed in terms of the 3rd Jacobi theta function

$$\vartheta_3(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau n^2 + 2in z} \quad (8)$$

by which we have [15]

$$\langle \phi_N, t_N | \phi_0, t_0 \rangle = \frac{1}{2\pi} \vartheta_3 \left(\frac{\phi_N - \phi_0}{2}, \frac{-\hbar(t_N - t_0)}{2\pi m R^2} \right) \quad (9)$$

Using the modular property of ϑ_3

$$\vartheta_3(z, \tau) = (-i\tau)^{-1/2} e^{-iz^2/\pi\tau} \vartheta_3 \left(\frac{z}{\tau}, -\frac{1}{\tau} \right) \quad (10)$$

the transition amplitude recasts as [15]

$$\langle \phi_N, t_N | \phi_0, t_0 \rangle = \sum_{n=-\infty}^{\infty} \sqrt{\frac{mR^2}{2\pi i \hbar(t_N - t_0)}} \exp \left(\frac{i}{\hbar} \frac{mR^2(\phi_N - \phi_0 - 2\pi n)^2}{2(t_N - t_0)} \right) \quad (11)$$

in which the summand is easily recognized as the transition amplitude of a free particle experiencing the position difference $x_N - x_0 = R(\phi_N - \phi_0 - 2\pi n)$ during time $t_N - t_0$ [14]. There is a nice interpretation for the sum as well. As the particle moves on the circle from ϕ_0 to ϕ_N , it matters how many times it rounds the circle. The sum on n , the so-called winding number, is responsible for taking into account the contributions from different rounds to the amplitude. So, nevertheless the coordinate ϕ has finite extent, practically the particle may travel long distances $\Delta x = R|\phi_N - \phi_0 - 2\pi n|$ for $n = 0, \pm 1, \pm 2, \dots$. In fact (11) may come to a form similar to (4) with only an extra summation. First let us present a time-sliced form of (11) [15]:

$$\begin{aligned} \langle \phi_N, t_N | \phi_0, t_0 \rangle &= \lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} \prod_{j=1}^{N-1} \sqrt{\frac{mR^2}{2\pi i \hbar \epsilon}} d\phi_j \\ &\times \prod_{l=0}^{N-1} \sum_{n_l=-\infty}^{\infty} \exp \left[\frac{i}{\hbar} \frac{mR^2(\phi_{l+1} - \phi_l - 2\pi n_l)^2}{2\epsilon} \right] \end{aligned} \quad (12)$$

It is noticed that at l -th time-slice the winding number n_l is introduced [15]. Now, by the following change in the integral variables [15]:

$$\sum_{n_l=-\infty}^{\infty} \int_{(2n_l-1)\pi}^{(2n_l+1)\pi} d\phi_l \rightarrow \int_{-\infty}^{\infty} d\Phi_l \quad (13)$$

the expression (12) recasts to [15]:

$$\begin{aligned} \langle \phi_N, t_N | \phi_0, t_0 \rangle &= \lim_{N \rightarrow \infty} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} \sqrt{\frac{mR^2}{2\pi i \hbar \epsilon}} d\Phi_j \\ &\times \exp \left[\frac{i}{\hbar} \sum_{i=0}^{N-1} \frac{mR^2(\Phi_{i+1} - \Phi_i)^2}{2\epsilon} \right] \end{aligned} \quad (14)$$

in which $\Phi_N = \phi_N$ and $\Phi_0 = \phi_0 + 2\pi n$. In above all the intermediate angles Φ_j 's are integrated over the whole real-line \mathbb{R} . In fact by the change of variable (13), integrating the intermediate angle ϕ_l over $[-\pi, \pi]$ together with sum over infinite possible rounds n_l is replaced by integration of Φ_l over $(-\infty, \infty)$.

The above example shows that it is insufficient to merely use finite-extent coordinates to be granted the non-trivial aspects. The key point about the above treatment of motion on circle is that the time-sliced form of the action appearing in the path-integral (12) is not invariant under the multi-round shifts

$$\phi_l \rightarrow \phi_l + 2\pi k_l, \quad k_l = 0, \pm 1, \pm 2, \dots \quad (15)$$

with $k_l \neq k_j$ for $l \neq j$. It is noticed that the above shift is directly related to the different number of possible rounds on circle represented earlier by n_l in (12). Reminding that the group space of $U(1)$ is circle, we can express the main idea in a group theoretical language. Defining the $U(1)$ group element by $U_\phi = \exp(i\phi)$, the action by the Lagrangian (5) can be expressed as [15]

$$S = \frac{1}{2}mR^2 \int dt \left(U_\phi^\dagger \dot{U}_\phi \right) \left(U_\phi^\dagger \dot{U}_\phi \right)^\dagger = \frac{1}{2}mR^2 \int dt \dot{\phi}^2 \quad (16)$$

In above, although the starting point is taken to be the group element U_ϕ , the Lagrangian depends in fact on the algebra element ϕ . This observation for the group $U(1)$ is general and holds for other groups as well [15]. Obviously the situation changes if the action would be invariant under the shift (15), namely due to dependence on group elements $\exp(i\phi_l)$'s instead of the algebra elements ϕ_l 's. Interestingly, once the time parameter is assumed to be discrete the desired dependence is obtained. By taking time as $t_n = na$ for some finite value a and integer n , the worldline looks like a chain or 1D lattice with spacing parameter a . On the n -th site of this chain the angle ϕ_n is sitting. So at time step n we have $U_n = \exp(i\phi_n)$, by which the discrete-time version of action (16) is

$$\begin{aligned} S &= \frac{1}{2}mR^2 a^{-1} \sum_n \left(U_n^\dagger (U_{n+1} - U_n) \right) \left(U_n^\dagger (U_{n+1} - U_n) \right)^\dagger \\ &= \frac{1}{2}mR^2 a^{-1} \sum_n (U_n^\dagger U_{n+1} - 1)(U_{n+1}^\dagger U_n - 1) \\ &= -\frac{1}{2}mR^2 a^{-1} \sum_n (U_n^\dagger U_{n+1} + U_{n+1}^\dagger U_n - 2) \end{aligned} \quad (17)$$

which obviously keeps invariance under the shift (15). The phenomena observed here is partially in reverse direction of what has happened in lattice formulation of gauge theories. Namely, once one tries to introduce gauge symmetry to the theory on lattice the algebra elements A_μ 's are to be replaced by the group elements $\exp(i a A_\mu)$'s in the action [4]. Here as we were going to keep the invariance under the shift (15) the natural solution appears to be defining the action on discrete-time worldline. In the next sections we will use this as a basis for model building.

Before to end this section it is helpful to discuss the prominent role of the imaginary time in the quantization of models with compact domain support in the sense described in above. In particular let us consider the matrix element by (4) [14]

$$\langle x | \hat{U} | x' \rangle = \langle x | \exp(-i \Delta t \hat{H} / \hbar) | x' \rangle = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} e^{i S[t, t + \Delta t] / \hbar} \quad (18)$$

in which \hat{U} is the unitary time evolution operator, \hat{H} is the Hamiltonian, and $S[t, t + \Delta t]$ is the action between times t and $t + \Delta t$ [14]. The basic observation is that the above representation in terms of the action is not possible when the dynamical variables are to take values inside a compact domain in the sense mentioned earlier. The reason can be easily understood for a system with one dynamical variable λ (field or coordinate). The generalization to systems with more variables is then straightforward. The identity $\hat{U} \hat{U}^\dagger = \hat{1}$ for the unitary evolution operator \hat{U} , defining $U(\lambda, \lambda'') = \langle \lambda | \hat{U} | \lambda'' \rangle$, in the λ -basis takes the form:

$$\int_{\Lambda} d\lambda'' U(\lambda, \lambda'') U^*(\lambda', \lambda'') = \delta(\lambda - \lambda') \quad (19)$$

in which Λ is the compact domain in which λ takes values. Now, by the representation like (19), as the integrand consists of only the ordinary regular functions and not the distribution ones, there is no way that the integral over a compact domain can develop a δ -function. By lacking the representation (18) for a unitary time evolution operator, the alternative is to assume that time is imaginary ($t \rightarrow -i t$). In the models with discrete time ($\Delta t = a$), by this alternative option the one-step unitary operator $\hat{U}_1 = \exp(-i a \hat{H} / \hbar)$ is replaced by the so-called transfer-matrix operator \hat{V} whose matrix element between two adjacent times n and $n + 1$ is given by [13]

$$\langle \lambda_{n+1} | \hat{V} | \lambda_n \rangle \propto \sqrt{m} \exp(S_E(n, n + 1) / \hbar) \quad (20)$$

in which $S_E(n, n+1)$ is the Euclidean action. Then by common eigen-states for \hat{H} and \hat{V} , the eigen-values of \hat{H} are defined by [4, 13]

$$E_s = -\hbar a^{-1} \ln v_s \quad (21)$$

where v_s is the corresponding eigen-value of \hat{V} . Provided that \hat{V} does not have negative eigen-values, the above would give a consistent description of the quantum theory based on an action with discrete imaginary time [4, 13]. This approach is exactly what is chosen in lattice formulation of gauge theories [4], turning space-time to a Euclidean one, and it will be used in the present work as well.

In summary, by the considerations mentioned in above, the followings are the basis for model building:

1. Time is assumed to be discrete and imaginary, taking values $t_n = n a$ for integer n .
2. The action with discrete time depends on the group elements to enhance the features by the compact nature of group.

3 U(1) Group

For the i -th direction with coordinate $-\pi R^i \leq x^i \leq \pi R^i$, the U(1) group element at n -th time step is taken as $U_n^i = \exp(i x_n^i / R)$. Here for simplicity we take all radii R^i 's equal to R . Following (17) the Euclidean action takes the form

$$\begin{aligned} S_E &= \frac{\kappa}{2} \sum_{n,i} (U_{n+1}^i U_n^{i*} + U_n^i U_{n+1}^{i*} - 2) \\ &= \kappa \sum_{n,i} \left(\cos \frac{x_{n+1}^i - x_n^i}{R} - 1 \right) \end{aligned} \quad (22)$$

The dimensionless constant κ is the defining parameter of the model (we have set $\hbar = c = 1$). The above, as discussed in previous section, is invariant under the shift:

$$x_n^i \rightarrow x_n^i + 2\pi k_n^i R, \quad (23)$$

with k_n^i 's as integer numbers. At the first place let us check the limits:

$$\begin{aligned} x/R &\ll 1 \\ x_{n+1} - x_n &\rightarrow a \dot{x} \\ \sum_n &\rightarrow a^{-1} \int dt \end{aligned} \quad (24)$$

leading to

$$S_E \simeq -\frac{a\kappa}{2R^2} \int dt \dot{x}_i^2 \quad (25)$$

which describes the dynamics of an ordinary free particle with mass $m_0 = a\kappa/R^2$ in the imaginary time formalism. As mentioned in Introduction, the action (22) is used in [11] to model the dynamics of 0-branes at strong coupling limit. The action (22) is the result of dimensional reduction of U(1) lattice gauge theory along spstial directions, by setting:

$$\begin{aligned} \kappa &= 1/g^2 \\ a A^i &\rightarrow x^i/R \end{aligned} \quad (26)$$

The expression (22) for the action is also known as the 1D plane-rotator model of magnetic systems [12], although here it is interpreted as a discrete worldline equipped by the angle variable coordinates x^i 's. In this section we review the construction by [11]. It is useful to define the new variables

$$y^i = x^i/R \quad (27)$$

taking values in $[-\pi, \pi]$, by which the action (22) takes the form

$$S_0 = \kappa \sum_{n,i} (\cos(y_{n+1}^i - y_n^i) - 1) \quad (28)$$

As the action is fully separable for each direction, it is sufficient to consider only one copy, dropping the index i hereafter. As mentioned in Sec. 2, the action with discrete imaginary time can be used to define the quantum theory based on the transfer-matrix \hat{V} , defined by its matrix elements

$$\langle y_{n+1} | \hat{V} | y_n \rangle = \sqrt{\frac{\kappa}{2\pi}} \exp[\kappa (\cos(y_{n+1} - y_n) - 1)] \quad (29)$$

in which, recalling $m_0 \propto \kappa$, the normalization prefactor has to be inserted to match the propagator (18) (see also (20) [13])

$$\langle x_2, t_2 | x_1, t_1 \rangle \propto \sqrt{\frac{m_0}{2\pi}} \exp\left(\frac{-m_0(x_2 - x_1)^2}{(t_2 - t_1)}\right) \quad (30)$$

Using the identity for the modified Bessel function of the first kind:

$$\exp[\kappa \cos(y' - y)] = \sum_{s=-\infty}^{\infty} I_s(\kappa) e^{is(y' - y)} \quad (31)$$

we have for (29)

$$\langle y_{n+1} | \widehat{V} | y_n \rangle = \sum_{s=-\infty}^{\infty} \sqrt{\frac{\kappa}{2\pi}} e^{-\kappa} I_s(\kappa) e^{i s (y_{n+1} - y_n)} \quad (32)$$

by which one reads the normalized plane-wave

$$\psi_s(x) = \frac{1}{\sqrt{2\pi}} \exp(i s y), \quad -\pi \leq y \leq \pi \quad (33)$$

as eigen-function with the eigen-value

$$v_s(\kappa) = \sqrt{2\pi\kappa} e^{-\kappa} I_s(\kappa) \quad (34)$$

By the known properties of I_s -functions we have $v_s = \sqrt{2\pi\kappa} e^{-\kappa} I_s(\kappa) \geq 0$. This guaranties that the transfer-matrix method defined by (20) and (21) would lead to a consistent quantum theory. Also by $I_s(z) = I_{-s}(z)$ the spectrum is doubly degenerate for $s \neq 0$. The energy eigenvalues are found by (21) and (34)

$$E_s(\kappa) = -\frac{1}{a} \ln \left[\sqrt{2\pi\kappa} e^{-\kappa} I_s(\kappa) \right] \quad (35)$$

The behavior of above at the limit $\kappa \rightarrow \infty$ can be checked by the saddle point approximation for Bessel functions

$$I_s(\kappa) = \lim_{\kappa \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} dy \exp(\kappa \cos y + i s y) \simeq \frac{e^{\kappa}}{\sqrt{2\pi\kappa}} \exp\left(-\frac{s^2}{2\kappa}\right) \quad (36)$$

by which for (35) we obtain

$$E_s \simeq \frac{s^2}{2a\kappa} \quad (37)$$

matching the energy $E = p^2/(2m_0)$ of a free particle with momentum $p = s/R$ along the compact direction, and mass $m_0 = \kappa a/R^2$ by (25). So in the limit $\kappa \rightarrow \infty$ the spectrum approaches to that of an ordinary particle. For the intermediate κ the spectrum is discrete. In the limit $\kappa \rightarrow 0$, using

$$I_s(z) \simeq \frac{1}{s!} \left(\frac{z}{2}\right)^s, \quad z \ll 1 \quad (38)$$

we have

$$E_s = -(s + \frac{1}{2}) \frac{\ln \kappa}{a} + O(s \ln s) + O(\kappa) \quad (39)$$

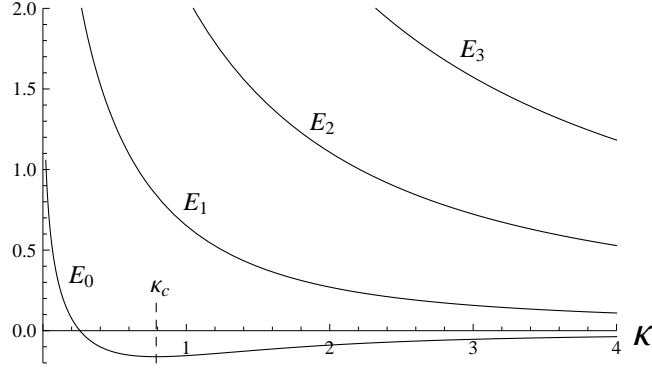


Figure 1: The few lowest energies by (35) versus κ (E unit: a^{-1}).

in which the 2nd term is independent of κ . Also at $\kappa \rightarrow 0$

$$E_{s+1} - E_s \simeq -\frac{\ln \kappa}{a} \gg \frac{1}{a} \quad (40)$$

The interesting observation by the spectrum (35) is about the energy of ground-state, which has a minimum at $\kappa_c = 0.790$; see Fig. 1. As expected the existence of minimum leads to a first order phase transition. The one-particle partition function may be evaluated by the definition

$$Z_1(\beta, \kappa) := \sum_{s=-\infty}^{\infty} e^{-\beta E_s(\kappa)} \quad (41)$$

or by means of the transfer-matrix operator (β in a units) [13]

$$Z_1(\beta, \kappa) = \text{Tr } \hat{V}^\beta = \int_{-\pi}^{\pi} \prod_{m=0}^{\beta-1} \sqrt{\frac{\kappa}{2\pi}} dy_m \exp \left[\kappa \sum_{n=0}^{\beta-1} (\cos(y_{n+1} - y_n) - 1) \right] \quad (42)$$

supplemented by the periodic condition $y_0 = y_\beta$. In the present case the equivalence of (41) and (42) is checked by numerical evaluations. The basic observation by the compact angle variable in above is, in contrast to the situation with infinite extent coordinates, the normalization factor can not be absorbed by a change of integration variable. As the minimum of E_0 is in variable κ , we need the thermodynamical conjugate variable M , defined by ($T = \beta^{-1}$)

$$M(\beta, \kappa) := T \frac{\partial \ln Z_1(\beta, \kappa)}{\partial \kappa} \quad (43)$$

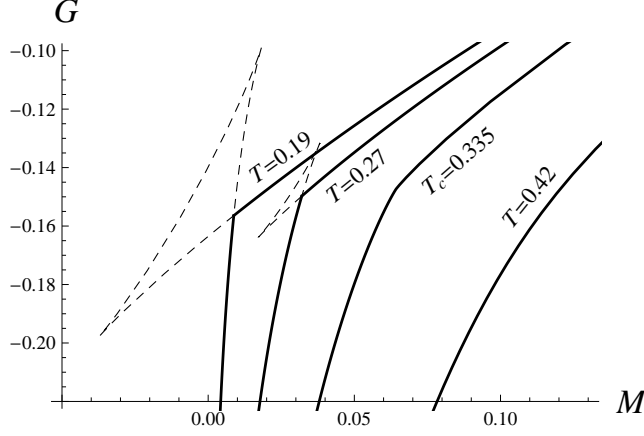


Figure 2: The G - M plots at four temperatures. The dashed pieces are not followed by the system due to the minimization of G .

which is also interpreted as the equation-of-state of the system. The Gibbs free energy can represent the exact nature of the phase transition,

$$G_1 = A_1 + \kappa M \quad (44)$$

in which $A_1 = -T \ln Z_1$ is the free energy per particle. The isothermal G - M plots are presented in Fig. 2. Evidently below the critical temperature $T_c = 0.335 a^{-1}$ the plots develop cusps, at which by the minimization of G at equilibrium, the system follows the path with lower G (solid-lines in Fig. 2). As the consequence, for $T < T_c$ there is a jump in first derivative of $\partial G / \partial M$, indicating that the phase transition is a first order one. It is apparent by now that the above phase structure is quite similar to the gas/liquid transition, for which G - P plots show exactly the same behavior. In a similar way the equation-of-state (43) should be modified by the so-called Maxwell construction for P - V diagram, by which during isothermal condensation the pressure (here M) is fixed. The results of the Maxwell construction for the present model are plotted as isothermal M - κ curves in Fig. 3. The flat part at T_c corresponds to values:

$$T_c = 0.335 : \quad \kappa^* = 1.403, \quad M^* = 0.064. \quad (45)$$

For isothermal curves below T_c , the straight horizontal parts describe the coexistent phases of lower and higher couplings during the phase transition. The interesting

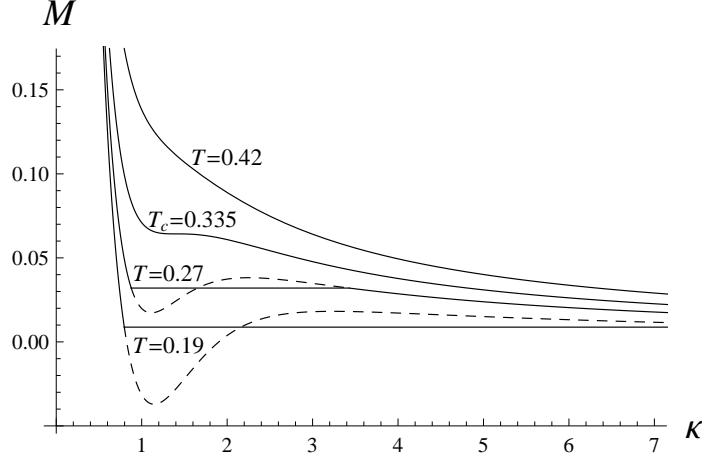


Figure 3: The isothermal M - κ plots. The straight-lines are due to the Maxwell construction, replacing the dashed parts.

fact about the equation-of-state modified by Maxwell construction is that M always remains non-negative, that is $M \geq 0$. This is specially important by expectations from the variable M at the limit $\kappa \gg 1$, at which we expect the ordinary behavior for particles. At this limit, back to (25) and (37), we have

$$M \simeq \frac{1}{2} \langle y^2 \rangle \propto \frac{T}{m_0} \quad (46)$$

where the proportionality is by the properties of free ordinary particles. In fact the asymptotic tails in Fig. 3 for large $m_0 \propto \kappa$ are explained by (46). There are also asymptotes at $\kappa \rightarrow 0$, although with different slopes. In fact the main difference between the case with present model and that of ordinary particles is about the existence of a phase transition. In particular, by the present model and below the critical temperature T_c , the two asymptotes by large and small masses (large and small κ 's) are connected with a first-order phase transition.

4 \mathbb{Z}_N Group

The coordinates with a compact domain may form a discrete group such as \mathbb{Z}_N . In this case the worldline looks like a spin chain with discrete spin degrees sitting on its sites. In general the worldline resemble the spin chain of Potts model (\mathbb{Z}_2 as of

the Ising model). The members of group \mathbb{Z}_N are presented by

$$\{1, \varrho, \varrho^2, \dots, \varrho^{N-1}\} \quad (47)$$

in which

$$\varrho = \exp(i 2\pi/N), \quad \varrho^N = 1 \quad (48)$$

At time-step n the position may be represented by r_n as

$$U_n = \varrho^{r_n} = \exp(i 2\pi r_n/N), \quad r_n = 0, 1, 2, \dots, N-1 \quad (49)$$

by which the action takes the form

$$\begin{aligned} S_E &= \frac{\kappa}{2} \sum_n (U_{n+1} U_n^* + U_n U_{n+1}^* - 2) \\ &= \kappa \sum_n \left(\cos \frac{2\pi(r_{n+1} - r_n)}{N} - 1 \right) \end{aligned} \quad (50)$$

The action is invariant under the shifts by k_n being any integer

$$r_n \rightarrow r_n + k_n N \quad (51)$$

It is convenient to define the new variable

$$\frac{2\pi r}{N} = w = 0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{(N-1)2\pi}{N} \quad (52)$$

by which the action comes to the form:

$$S_E = \kappa \sum_n (\cos(w_{n+1} - w_n) - 1) \quad (53)$$

The transfer-matrix element then easily reads

$$\langle n+1 | \widehat{V} | n \rangle = \frac{1}{N} \sqrt{2\pi\kappa} \exp[\kappa (\cos(w_{n+1} - w_n) - 1)] \quad (54)$$

$$= \sum_{s=0}^{N-1} e^{-a E_s} \frac{1}{N} e^{i s (w_{n+1} - w_n)} \quad (55)$$

in which the plane-waves

$$\psi_s(w) = \frac{1}{\sqrt{N}} e^{i s w} \quad (56)$$

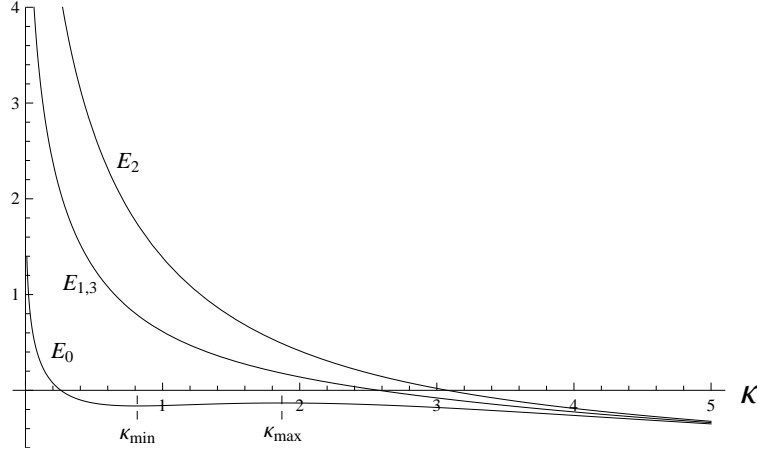


Figure 4: The energies by (59) for \mathbb{Z}_4 group.

satisfy the ortho-normality condition

$$\sum_w \psi_s^*(w) \psi_{s'}(w) = \delta_{ss'} \quad (57)$$

By the identity (31) for Bessel functions, and using the condition (57), the sum on Bessel functions can be partitioned into N cyclic ones, leading to

$$e^{-a E_s} = \sqrt{2\pi\kappa} e^{-\kappa} \sum_{q=-\infty}^{\infty} I_{s+qN}(\kappa) \quad (58)$$

in which the sum is converging due to properties by I_s -functions. So the energy eigen-values are simply given by

$$E_s(\kappa) = -a^{-1} \ln \left[\sqrt{2\pi\kappa} e^{-\kappa} \sum_{q=-\infty}^{\infty} I_{s+qN}(\kappa) \right] \quad (59)$$

for $s = 0, 1, 2, \dots, N-1$. Due to $I_s = I_{-s}$ we have the following degeneracy:

$$E_s = E_{N-s} \quad (60)$$

In fact the lowest and highest eigen-values are as follow

$$\begin{aligned} E_{\min} &= E_0, \\ E_{\max} &= E_{N/2}, \quad N : \text{even} \\ E_{\max} &= E_{(N\pm 1)/2}, \quad N : \text{odd} \end{aligned} \quad (61)$$

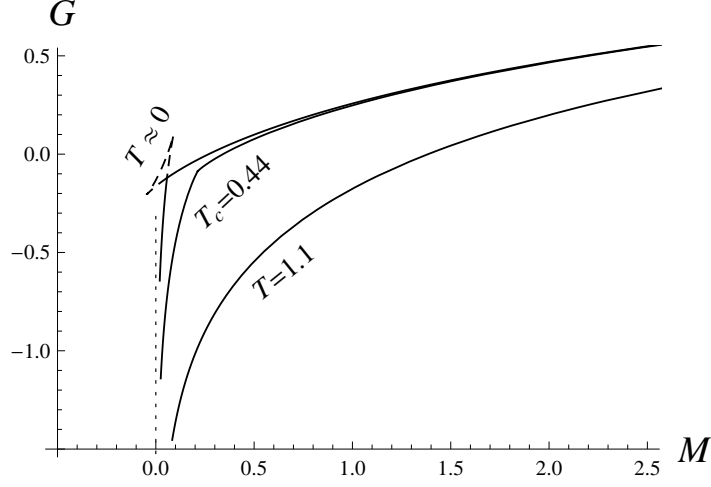


Figure 5: The G - M plots for \mathbb{Z}_4 group.

In the limit $N \rightarrow \infty$ we expect to recover the spectrum (35) by $U(1)$ group. It is in fact the case using

$$\frac{I_{\pm\infty}}{I_{\text{finite}}} \rightarrow 0 \quad (62)$$

It can easily be checked that for $N = 2$ & 3 there is no extrema in E_0 . For $N \geq 4$ there are both minimum and maximum. The energy eigen-values are plotted in Fig. 4 for the group \mathbb{Z}_4 , in which the extrema are at

$$\kappa_{\min} = 0.815, \quad \kappa_{\max} = 1.87 \quad (63)$$

Again the one-particle partition function (41), together with the thermodynamical functions (43) and (44), can be defined for the system. Like the case with $U(1)$ group, the appearance of minimum leads to a phase transition. The isothermal G - M plots for the \mathbb{Z}_4 are plotted in Fig. 5. As expected below the critical temperature $T_c = 0.44$ the plots develop cusps, exhibiting a first order phase between two co-existing phases with low and high κ 's. However, at $T \approx 0$ there is a difference between the \mathbb{Z}_N and $U(1)$ groups, that is here there is a finite higher κ at which the systems follows the path with finite M 's. This behavior is evident by $T \approx 0$ curve in Fig. 6, in contrast to $T \approx 0$ curve for $U(1)$ group in Fig. 4, with $M = 0$ toward $\kappa \rightarrow \infty$. The two critical κ 's of $T \approx 0$ for \mathbb{Z}_4 group at which the cusp starts and ends are as follow

$$\kappa_{c1} = 0.69, \quad \kappa_{c2} = 8.8 \quad (64)$$

5 SU(2) Group

As the case for a non-Abelian group here we consider the SU(2) group in one spatial direction. Then at time-step n the group element is represented by

$$U_n = \exp(i \mathbf{x}_n \cdot \boldsymbol{\sigma} / 2R) \quad (65)$$

in which $\mathbf{x}_n = (x_n^1, x_n^2, x_n^3)$ represents the three components in the SU(2) sector, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ as Pauli matrices. The action then simply comes to the form

$$S_E = \frac{\kappa}{2} \sum_n \text{Tr} \left(U_{n+1} U_n^\dagger + U_n U_{n+1}^\dagger - 2 \mathbb{1}_2 \right) \quad (66)$$

in which Tr is trace over the matrix structure, with $\text{Tr}(\sigma_\alpha \sigma_\beta) = 2 \delta_{\alpha\beta}$. As the requirement mentioned earlier, the above action is invariant under the shift:

$$|\mathbf{x}_n| \rightarrow |\mathbf{x}_n| + 4\pi k_n R, \quad (67)$$

where k_n 's as integer numbers. Using the identity

$$\begin{aligned} \frac{1}{2} \text{Tr} (e^{-i \mathbf{x}_{n+1} \cdot \boldsymbol{\sigma} / 2R} e^{i \mathbf{x}_n \cdot \boldsymbol{\sigma} / 2R}) &= \cos \frac{r_{n+1}}{2R} \cos \frac{r_n}{2R} + \hat{\mathbf{x}}_{n+1} \cdot \hat{\mathbf{x}}_n \sin \frac{r_{n+1}}{2R} \sin \frac{r_n}{2R} \\ &=: \cos \frac{\gamma_{n+1,n}}{2} \end{aligned} \quad (68)$$

in which $r_n = |\mathbf{x}_n|$ and $\hat{\mathbf{x}}_n = \mathbf{x}_n / r_n$, the actions is simplified as

$$S_E = 2\kappa \sum_n \left(\cos \frac{\gamma_{n+1,n}}{2} - 1 \right) \quad (69)$$

In the limits r_n & $r_{n+1} \ll R$ we have

$$\gamma_{n+1,n}^2 \simeq \frac{1}{R^2} (\mathbf{x}_{n+1} - \mathbf{x}_n)^2 + O\left(\frac{r}{R}\right)^4 \quad (70)$$

by which we have in the continuum limit

$$S_E \simeq -\frac{a\kappa}{4R^2} \int dt \dot{\mathbf{x}}^2 \quad (71)$$

In above again the minus sign is due to use of imaginary time in the formalism. The action (71) represents the free motion of a free particle with mass $m_0 = a\kappa / (2R^2)$.

As before, the action (69) can be used to define the quantum theory based on the transfer-matrix method. The group manifold of SU(2) is known to be the 3-sphere

S^3 , for which the initial parametrization $\mathbf{x} = (x_1, x_2, x_3)$ in the spherical coordinates $\mathbf{x} = (r, \theta, \phi)$ is defined in the intervals

$$0 \leq r \leq 2\pi R, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi \quad (72)$$

Also it is convenient to use the replacement

$$\chi = \frac{r}{2R}, \quad 0 \leq \chi \leq \pi \quad (73)$$

By the above parametrization the $SU(2)$ invariant measure takes the form [16]

$$d\Omega_3 = \sin^2 \chi d\chi d\Omega, \quad \text{with} \quad d\Omega = \sin \theta d\theta d\phi \quad (74)$$

satisfying

$$\int_{S^3} d\Omega_3 = 2\pi^2 \quad (75)$$

As the case with $U(1)$ group, by the complete orthonormal spherical harmonics on S^3 as the eigen-functions, we can read the eigen-values of the matrix \hat{V} . The normalized spherical harmonics on S^3 are known to be [16]

$$\mathbb{Y}_{s\ell m}(\chi, \Omega) = \sqrt{\frac{2^{2\ell+1}(s+1)(s-\ell)! \ell!^2}{\pi(s+\ell+1)!}} \sin^\ell \chi C_{s-\ell}^{(\ell+1)}(\cos \chi) Y_{\ell m}(\Omega) \quad (76)$$

in which $C_{s-\ell}^{(\ell+1)}(x)$ are the Gegenbauer polynomials and $Y_{\ell m}(\Omega) = Y_{\ell m}(\theta, \phi)$ are the ordinary spherical harmonics on 2-sphere S^2 . In above all indices are integers and obey the ordering [16]

$$|m| \leq \ell \leq s = 0, 1, 2, \dots \quad (77)$$

The above harmonics are the normalized ones [16]

$$\int_{S^3} d\Omega_3 \mathbb{Y}_{s\ell m}(\chi, \Omega) \mathbb{Y}_{s'\ell'm'}^*(\chi, \Omega) = \delta_{ss'} \delta_{\ell\ell'} \delta_{mm'} \quad (78)$$

Using (68) and in the (χ, θ, ϕ) parametrization of S^3 , the action (69) between the adjacent times n and $n+1$ takes the form

$$S_E(n, n+1) = 2\kappa \left(\cos \chi_{n+1} \cos \chi_n + \hat{\mathbf{x}}_{n+1} \cdot \hat{\mathbf{x}}_n \sin \chi_{n+1} \sin \chi_n - 1 \right) \quad (79)$$

in which

$$\hat{\mathbf{x}}_{n+1} \cdot \hat{\mathbf{x}}_n = \cos \theta_{n+1} \cos \theta_n + \cos(\phi_{n+1} - \phi_n) \sin \theta_{n+1} \sin \theta_n \quad (80)$$

Then the matrix elements of the transfer matrix is given by

$$\langle \mathbf{x}_{n+1} | \hat{V} | \mathbf{x}_n \rangle = \sqrt{\frac{\kappa}{4\pi}} \exp \left[S_E(n, n+1) \right] \quad (81)$$

Using the identity for real w

$$e^{w \hat{\mathbf{x}} \cdot \hat{\mathbf{x}}'} = \sqrt{\frac{\pi}{2w}} 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} I_{\ell+1/2}(w) Y_{\ell m}(\Omega) Y_{\ell m}^*(\Omega') \quad (82)$$

in which the direction of two unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ are given by ordinary solid-angles Ω and Ω' , respectively, and $I_{\ell+1/2}$ as before is the modified Bessel function. Taking $w = 2\kappa \sin \chi_n \sin \chi_{n+1}$, by (82) we have

$$\begin{aligned} \sqrt{\frac{\kappa}{4\pi}} \exp \left[S_E(n, n+1) \right] &= \sqrt{\frac{\kappa}{4\pi}} e^{2\kappa(\cos \chi_n \cos \chi_{n+1} - 1)} \sqrt{\frac{\pi}{4\kappa \sin \chi_n \sin \chi_{n+1}}} \\ &\times 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} I_{\ell+1/2}(2\kappa \sin \chi_n \sin \chi_{n+1}) Y_{\ell m}(\Omega_n) Y_{\ell m}^*(\Omega_{n+1}) \end{aligned} \quad (83)$$

for which we also have an expansion based on the energy eigen-values and the eigen-functions as

$$\begin{aligned} \sqrt{\frac{\kappa}{4\pi}} \exp \left[S_E(n, n+1) \right] &= \sum_{s=0}^{\infty} \sum_{\ell=0}^s \sum_{m=-\ell}^{\ell} e^{-aE_{s\ell m}} \mathbb{Y}_{s\ell m}(\chi_n, \Omega_n) \mathbb{Y}_{s\ell m}^*(\chi_{n+1}, \Omega_{n+1}) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{s=\ell}^{\infty} e^{-aE_{s\ell m}} \mathbb{Y}_{s\ell m}(\chi_n, \Omega_n) \mathbb{Y}_{s\ell m}^*(\chi_{n+1}, \Omega_{n+1}) \end{aligned} \quad (84)$$

Using the orthonormality relation of $Y_{\ell m}$'s, by (83) and (84), after the changes $\chi_n \rightarrow \chi$ and $\chi_{n+1} \rightarrow \chi'$, explicit expression (76) gives us

$$\begin{aligned} \sum_{s=\ell}^{\infty} \frac{2^{2\ell+1} (s+1) (s-\ell)! \ell!^2}{\pi (s+\ell+1)!} e^{-aE_{s\ell}} \sin^{\ell} \chi \sin^{\ell} \chi' C_{s-\ell}^{(\ell+1)}(\cos \chi) C_{s-\ell}^{(\ell+1)}(\cos \chi') \\ = \sqrt{\frac{\kappa}{4\pi}} \sqrt{\frac{\pi}{4\kappa \sin \chi \sin \chi'}} 4\pi e^{2\kappa(\cos \chi \cos \chi' - 1)} I_{\ell+1/2}(2\kappa \sin \chi \sin \chi') \end{aligned} \quad (85)$$

in which we have dropped the index m in $E_{s\ell}$, as it is now an irrelevant one. This

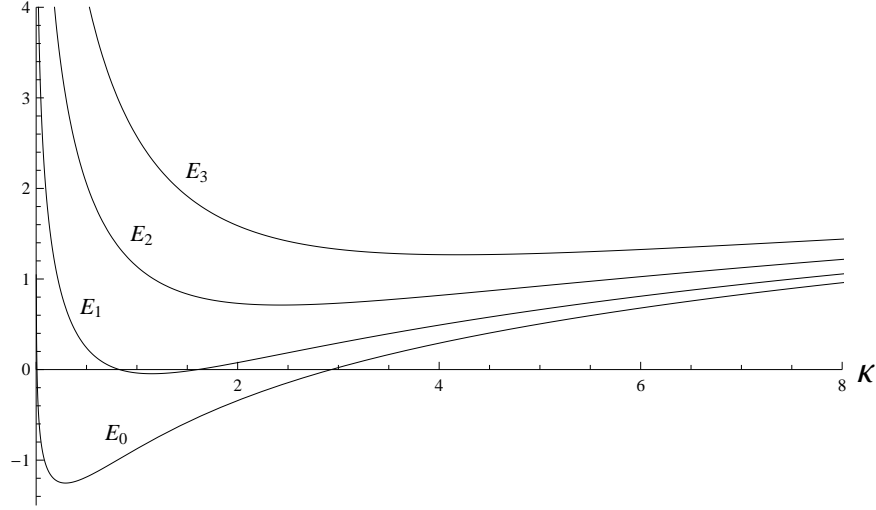


Figure 6: The few lowest energies by (90) for SU(2) group.

indicates that the energy eigen-values has at least $(2\ell + 1)$ -level degeneracy. Using the orthogonality of the Gegenbauer polynomials

$$\int_0^\pi C_{s-\ell}^{(\ell+1)}(\cos \alpha) C_{s'-\ell}^{(\ell+1)}(\cos \alpha) \sin^{2\ell+2} \alpha d\alpha = \frac{\pi (s + \ell + 1)!}{2^{2\ell+1} (s + 1) (s - \ell)! \ell!^2} \delta_{ss'} \quad (86)$$

by multiplication the l.h.s. of (85) by $\sin^{\ell+2} \chi' \cos_{s-\ell}^{(\ell+1)}(\cos \chi')$ and integration over χ' we have

$$e^{-aE_{s\ell}} \sin^\ell \chi C_{s-\ell}^{(\ell+1)}(\cos \chi) = \int_0^\pi d\chi' \sqrt{\frac{\kappa}{4\pi}} \sqrt{\frac{\pi}{4\kappa \sin \chi \sin \chi'}} 4\pi e^{2\kappa(\cos \chi \cos \chi' - 1)} \times \sin^{\ell+2} \chi' \cos_{s-\ell}^{(\ell+1)}(\cos \chi') I_{\ell+1/2}(2\kappa \sin \chi \sin \chi') \quad (87)$$

Using the identity [17]:

$$\begin{aligned} \int_0^\pi e^{z \cos \chi \cos \chi'} \sin^{\ell+3/2} \chi' C_{s-\ell}^{(\ell+1)}(\cos \chi') I_{\ell+1/2}(z \sin \chi \sin \chi') d\chi' \\ = \sqrt{\frac{2\pi}{z}} \sin^{\ell+1/2} \chi C_{s-\ell}^{(\ell+1)}(\cos \chi) I_{s+1}(z) \end{aligned} \quad (88)$$

the integration in r.h.s. of (87) can be calculated, setting $z = 2\kappa$, leading to

$$e^{-aE_s} = \sqrt{\frac{\kappa}{4\pi}} \frac{2\pi^2}{\kappa} e^{-2\kappa} I_{s+1}(2\kappa) \quad (89)$$

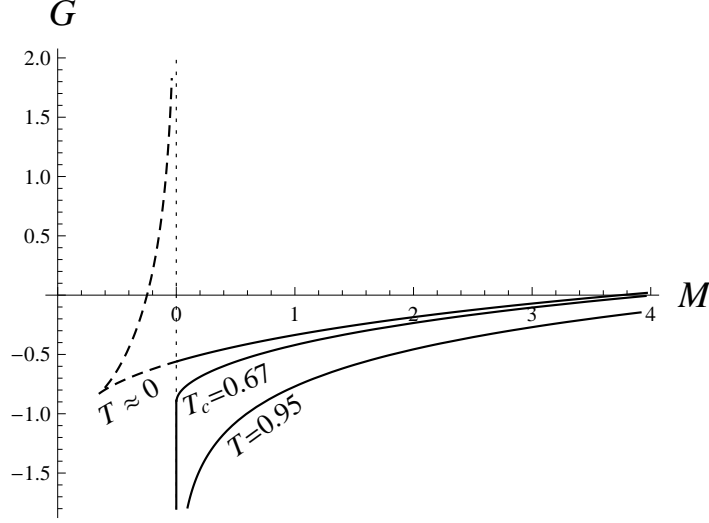


Figure 7: The isothermal G - M plots for $SU(2)$ group.

in which we have dropped the index l from E_s as well, since there is no ℓ -dependence in spectrum. By (89) we find the energy as below with a $(s+1)^2$ -level degeneracy:

$$E_s = -\frac{1}{a} \ln \left[\sqrt{\frac{\kappa}{4\pi}} \frac{2\pi^2}{\kappa} e^{-2\kappa} I_{s+1}(2\kappa) \right], \quad s = 0, 1, 2, \dots \quad (90)$$

In Fig. 6 the four lowest energies are plotted, all having minimum. This is in contrast to the cases with $U(1)$ and \mathbb{Z}_N groups where only E_0 has a minimum. The one-particle partition function is given by

$$Z_1(\beta, \kappa) = \sum_{s=0}^{\infty} (s+1)^2 e^{-\beta E_s} \quad (91)$$

Again the equation-of-state as well as the Gibbs free energy can be obtained by (43) and (44). The G - M plots given in Fig. 7 develop cusp, indicating that the system exhibits a first-order phase transition below the critical temperature $T_c = 0.67$. At $T \approx 0$ the critical $\kappa_c = 0.29$ is obtained above which M gets non-zero values.

6 Conclusion and Discussion

The general theoretical ground as well as specific examples are presented for models based on compact angle coordinates. The present construction might be considered

as a continuation of the theme by which the gauge fields are treated as compact angle variables [1–4]. In the present formulation the action depends on the group elements rather than the algebra elements, leading to the invariance under the total shifts inside the compact domain. It is observed that a discrete-time formulation of the theory is the natural way to obtain the desired dependence on group elements. The way toward the present formulation is in some sense in reverse direction of what has happened in lattice formulation of gauge theories. In particular, in formulation of gauge theories on lattice the invariance under the gauge transformations requires that the gauge fields are introduced to the theory via the group elements. Here the demand of invariance under the total shifts of compact coordinates requires to treat time parameter as a discrete one. The worldline action by the formulation resembles the spin chain Hamiltonian of magnetic systems, with coordinates appearing as the spin degrees of freedom.

The quantization of the model is formulated based on the transfer-matrix method [4, 13]. The pre-factor in the definition of the elements of transfer-matrix, in contrast to the case with infinite extent coordinates, can not be absorbed by a change of the path-integral integration variables. This particularly causes that the energy eigenvalues develop minima as functions of the defining parameter of the theory.

As examples for the formulation, the models based on the $U(1)$, \mathbb{Z}_N and $SU(2)$ groups are explicitly constructed. In all of the models based on the three groups the exact energy eigenvalues are obtained. As the consequence of the minima in the spectrum all the models exhibit the first-order transition between coexistent phases.

As mentioned earlier, by setting $\kappa = 1/g^2$ the model by $U(1)$ group is in fact the result of the dimensional reduction of pure $U(1)$ lattice gauge theory. As one possible application of the present construction here we mention the attempt in [11] to fit the model by $U(1)$ group to the expectations from the monopole dynamics. In particular, the phase transition for the particles with mass $m_0 \propto 1/g^2$ by (25) may provide a better understanding of the role of monopoles in confinement mechanism based on the dual Meissner effect in superconductors [18–20]. Based on the proposed mechanism, at strong coupling limit, at which the monopoles have tiny masses, the motion of monopoles around the electric fluxes prevents the fluxes to spread, leading to the confinement of the electric charges. Instead at small coupling limit, where the monopoles are highly massive, the electric fluxes originated from source charges are likely to spread over space, leading to the Coulomb’s law. It is expected that there

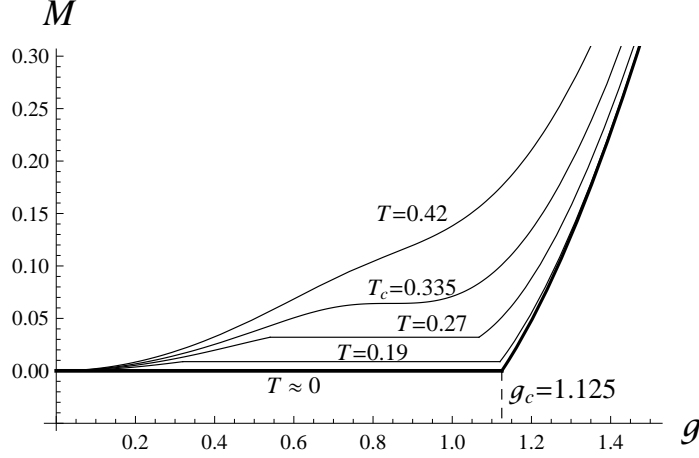


Figure 8: The isothermal M - g plots by the model based on the $U(1)$ group.

is a critical coupling g_c at which the transition from confined phase to the Coulomb phase occurs.

According to the model with group $U(1)$, the two regimes with weak and strong couplings constants are related by a first-order phase transition. The behavior of the system at low temperatures, where the main contribution to the partition function is from the ground-state, is of particular interest. In the limit $T \rightarrow 0$, due to the Maxwell construction, we have $M = 0$ for $g < g_c = 1.125$; Fig. 8. So as the consequence of discontinuous nature of the first order transition, at low temperatures and below g_c we have $M \propto \langle v^2 \rangle \approx 0$. This behavior is to be compared with (46) for ordinary particles, by which there is an asymptote reduce of M by increasing the mass at constant T . According to the present model, at low temperatures and below g_c , the particles with mass $m_0 \propto g^{-2}$ are hardly moving ($\langle v^2 \rangle \approx 0$), leading to an exact Coulomb phase. On the other hand, exhibiting a high-slope increase of $\langle v^2 \rangle$ at g_c , the confined phase finds an instant govern once g exceeds g_c at low temperatures. This picture and specially the value of critical coupling constant are in agreement with theoretical and numerical studies [11].

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